When can you play positionnaly?

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Abstract. We consider infinite antagonistic games over finite graphs. We present conditions that, whenever satisfied by the payoff mapping, assure for both players positional (memoryless) optimal strategies. We show that all popular payoff mappings, such as mean payoff, discounted, parity as well as several other payoffs satisfy these conditions. This approach allows to give a uniform treatment of otherwise disparate results concerning the existence of positional optimal strategies.

1 Introduction

We study antagonistic games played on finite oriented graphs G by two players Max and Min. Each vertex of G belongs to one of the players. If the current game position is a vertex v then the owner of v chooses an outgoing edge e and the target of e becomes a new game position. After an infinite number of moves we obtain an infinite path p in G that we call a play.

We suppose that the edges of G are coloured by elements of some set C. Then each play yields an infinite sequence of colours of the edges traversed during the play. The payoff functions indicates for each such infinite sequence of colours a real number: the amount that player Min pays to player Max at the end of the game.

This types of games is studied since many years in game theory even in much more general setting of stochastic games[9].

In general optimal strategies of both players can depend on the whole past history. However it turns out that for many games both players have optimal positional strategies, i.e. optimal strategies where the players' moves depend only on the current position. This type of strategies is particularly interesting in computer science since positional strategies allow an easy and efficient implementation.

Motivated by economic applications classical game theory studies mainly, but not exclusively, two payoff functions: mean-payoff and discounted [6]. Since the seminal paper of Shapley [9] it is known that even for stochastic discounted games both players have positional optimal strategies.

The existence of optimal positional strategies for mean-payoff deterministic games was established by Mycielski and Świerczkowski[4].

Let us note that recently discounted and even mean payoff games entered also in computer science, see [3,2] where are also nicely exposed the motivations behind discounting system properties.

Parity games have have much more recent history. They appear in the work of Emerson and Jutla [5] in relation to the modal μ-calculus and in Mostowski [7] in relation to the problem of determinizing finite automata over infinite trees (Rabin theorem). Again these games admit optimal positional strategies (even over infinite graphs).

As noted recently by Björklund et al. [1] it is possible to present highly similar proofs of the existence of optimal positional strategies for mean-payoff and parity games. Let us note however that [1] failed to extract explicitly the ingredients of both proofs that make them so similar. One can only guess that there are some common axioms hidden in the proof. Moreover the inductive method presented in [1] fails for discounted games.

In this paper we return to the problem of unifying all, somehow disparate, results of the existence of positional optimal strategies. We present three conditions that, when satisfied by payoff mapping, assure the existence of optimal positional strategies for both players. In the second part of the paper we show that virtually all payoff mappings that allow positional optimal strategies, including discounted payoff where the method of [1] failed, satisfy our conditions. This provides a clear explanation why optimal positional strategies are so omnipresent.

Let us note finally that the inductive method developed in our paper was successfully adapted by the second author to perfect information stochastic games where it allowed to show in a simpler way the existence of optimal positional pure (i.e. deterministic) strategies [10].

An intriguing open question is whether our three conditions assure the existence of optimal positional pure strategies for perfect information stochastic games in general.

2 Preliminaries

For any (possibly infinite) set C, we write C^+ to denote the set of all finite non empty words over C. An infinite word $c_0c_1c_2\ldots$ over C is said to be *finitely generated* if there is a finite subset A of C such that for all i, $c_i \in A$. In our paper C^ω will stand always for a set of all *finitely generated infinite words* over C. Let us note that this is a departure from the standard notation where C^ω stands usually for the set of all infinite sequences (infinite words) over C. The difference appears of course only if C is infinite. However, in our paper while it is usuful to allow infinite alphabets, only finitely generated sequences are of interest.

For $x \in C^+$, by $x^\omega = xxx\ldots$ we note the infinite concatenation of words x. We use also the standard mathematical notation: for any sequence $a_n, n \geq 0$, of real numbers, $\limsup_n a_n = \lim_{n\to\infty} \sup_{i\geq n} a_i$.

$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ will denote the set of reals extended with infinity values.

An *arena* is a tuple

$$G = (V_{\text{Max}}, V_{\text{Min}}, E, C, \varphi),$$

where $(V_{\text{Max}} \cup V_{\text{Min}}, E)$ is a finite oriented graph with the set $V = V_{\text{Max}} \cup V_{\text{Min}}$ of vertices partitioned onto the set V_{Max} of vertices of player Max and the set V_{Min} of vertices belonging to player Min. $E \subseteq V \times V$ is the set of edges. We shall colour edges by means of a mapping $\varphi : E \to C$ which associates with each edge $e \in E$ a colour $\varphi(e) \in C$. Although the set of colours will be often infinite (for example $\mathbb{R}$ or $\mathbb{N}$) actually only finite subsets of C will be used since we restrict our attention to finite arenas.

For any edge $e = (v, w) \in E$ we note $\text{source}(e) = v$ the source and $\text{target}(e) = w$ the target of e. For a vertex $v \in V$, by $vE = \{e \in E \mid \text{source}(e) = v\}$ we denote the set of edges outgoing from v.

We suppose that arenas have no dead-ends, i.e. each vertex has at least one outgoing edge.

A path in G is a finite or infinite sequence of edges $p = e_0 e_1 e_2 \ldots$ such that, for all $i \geq 0$, $\text{target}(e_i) = \text{source}(e_{i+1})$. The source of p is the source of the first edge e_0. If p is finite then $\text{target}(p)$ is the target of the last edge in p. It is convenient to assume that for each vertex v there exists an empty path λ_v with no edges and such that $\text{source}(\lambda_v) = \text{target}(\lambda_v) = v$.

Players Max and Min play on G in the following way. If the current game position is a vertex $v \in V_{\text{Max}}$ then player Max choses an outgoing edge $e \in vE$ and vertex $\text{target}(e)$ becomes a new game position. Otherwise, if the current game position v belongs to player Min then Min chooses an outgoing edge $e \in vE$ and vertex $\text{target}(e)$ becomes a new game position. If the initial position was v then in this way the players construct an infinite path $\mathbf{e} = e_0 e_1 e_2 \ldots$ of visited edges such that $\text{source}(\mathbf{e}) = v$. Such an infinite path will be called sometimes a play in G. The set of all plays (infinite paths) in G will be denoted P_G^ω. The set of finite paths in G will be noted P_G^*. Elements of P_G^* will be sometimes called *histories* or finite plays, especially when they are used to encode the history of all movements of both players up to a current moment.

With any play $\mathbf{e} = e_0 e_1 e_2 \ldots$ we associate the payoff sequence

$$\varphi(\mathbf{e}) = \varphi(e_0)\varphi(e_1)\varphi(e_2)\ldots$$

of visited colours.

Note that we have extended in this way the colouring mapping to $\varphi : P_G^\omega \to C^\omega$. In a similar way, we set for a finite path $p = e_0 \ldots e_k$, $\varphi(p) = \varphi(e_0)\ldots\varphi(e_k)$.

A *payoff function* is a mapping

$$u : C^\omega \to \overline{\mathbb{R}}.$$

Intuitively, after an infinite play p player Min pays to player Max the amount $u(\varphi(p))$ (with the natural interpretation that if $u(\varphi(p)) < 0$ then it is rather the player Max that pays to player Min the amount $|u(\varphi(p))|$). Let us note that since our arenas are finite $\varphi(p)$ will be always finitely generated for any infinite path

p. This is important since in some rare cases of payoff mappings u the value $u(p)$ will not be even well defined for infinite words which are not finitely generated.

A *game* is a couple $\mathbf{G} = (G, u)$ made of an arena G and a payoff function.

Let G be an arena and $\mathfrak{p} \in \{\mathrm{Min}, \mathrm{Max}\}$ a player. A *strategy* for player $\mathfrak{p}$ in G is a mapping $\sigma_{\mathfrak{p}}$ which indicates for each finite history p an edge outgoing from $\mathrm{target}(p)$ that player $\mathfrak{p}$ should choose. Thus formally

$$\sigma_{\mathfrak{p}} : \{p \in P_G^* \mid \mathrm{target}(p) \in V_{\mathfrak{p}}\} \to E$$

where $\sigma_{\mathfrak{p}}(p) \in vE$ whenever $v = \mathrm{target}(p)$.

A finite or infinite path $\mathbf{e} = e_0 e_1 e_2 \ldots$ is said to be *consistent* with the strategy $\sigma_{\mathfrak{p}}$ of player $\mathfrak{p}$ if whenever $\mathrm{target}(e_i) \in V_{\mathfrak{p}}$ then $e_{i+1} = \sigma_{\mathfrak{p}}(e_0 ... e_i)$ and moreover $e_0 = \sigma_{\mathfrak{p}}(\lambda_v)$ where $v = \mathrm{source}(p)$.

$\Sigma_{\mathfrak{p}}$ denotes the set of strategies of a player $\mathfrak{p}$.

In this paper we are especially interested in the class of positional strategies.

A *positional* strategy for player $\mathfrak{p}$ is a mapping $\sigma_{\mathfrak{p}} : V_{\mathfrak{p}} \to E$ such that for all $v \in V_{\mathfrak{p}}$, $\sigma_{\mathfrak{p}}(v) \in vE$. Intuitively, when $\mathfrak{p}$ uses a positional strategy then his choice of the outgoing edge depends only on the current position and not on the previous history. Obviously, a positional strategy is a strategy in the previous sense, for a finite history $p = e_0 \ldots e_k$ with $\mathrm{target}(p) \in V_{\mathfrak{p}}$ it suffices to set $\sigma_{\mathfrak{p}}(p) := \sigma_{\mathfrak{p}}(\mathrm{target}(p))$.

Given a vertex v and strategies σ and τ for player Max and player Min respectively, there exists a unique play starting in v and consistent with σ and τ. This play is denoted by $p^G(v, \sigma, \tau)$.

Let us set

$$\begin{aligned} \underline{\mathrm{val}}(\mathbf{G}(v)) &= \sup\nolimits_{\sigma \in \Sigma_{\mathrm{Max}}} \inf\nolimits_{\tau \in \Sigma_{\mathrm{Min}}} u(\varphi(p^G(v, \sigma, \tau))) \\ \overline{\mathrm{val}}(\mathbf{G}(v)) &= \inf\nolimits_{\tau \in \Sigma_{\mathrm{Min}}} \sup\nolimits_{\sigma \in \Sigma_{\mathrm{Max}}} u(\varphi(p^G(v, \sigma, \tau))). \end{aligned}$$

Intuitively, player Max can assure himself the payoff of at least $\underline{\mathrm{val}}(\mathbf{G}(v))$ while player Min can assure that his loss will not be greater than $\overline{\mathrm{val}}(\mathbf{G}(v))$. The quantity $\underline{\mathrm{val}}(\mathbf{G}(v))$ is called the lower value of the game $\mathbf{G}$ at v while $\overline{\mathrm{val}}(\mathbf{G}(v))$ is the upper value of $\mathbf{G}$ at v. Note that always

$$\underline{\mathrm{val}}(\mathbf{G}(v)) \leq \overline{\mathrm{val}}(\mathbf{G}(v)).$$

If $\underline{\mathrm{val}}(\mathbf{G}(v)) = \overline{\mathrm{val}}(\mathbf{G}(v))$ then $\mathbf{G}$ is said to have a value at v and we note such a value $\mathrm{val}(\mathbf{G}(v))$.

Strategies $\sigma^\sharp \in \Sigma_{\mathrm{Max}}$ and $\tau^\sharp \in \Sigma_{\mathrm{Min}}$ are said to be *optimal* if for all vertices $v \in V$ and all strategies $\sigma \in \Sigma_{\mathrm{Max}}$ and $\tau \in \Sigma_{\mathrm{Min}}$

$$u(p^G(v, \sigma, \tau^\sharp)) \leq u(p^G(v, \sigma^\sharp, \tau^\sharp)) \leq u(p^G(v, \sigma^\sharp, \tau)). \tag{1}$$

If (1) holds then the value of $\mathbf{G}$ exists for any initial vertex and $\mathrm{val}(\mathbf{G}(v)) = u(p^G(v, \sigma^\sharp, \tau^\sharp))$.

3 Fairly mixing payoffs yield positional strategies

The aim of this section is to present sufficient conditions for the payoff mapping u assuring the existence of optimal positional strategies for both players.

Definition 1. *A payoff mapping $u : C^\omega \to \overline{\mathbb{R}}$ is said to be a fairly mixing if the following conditions hold*

(C1) for all $x \in C^+, y_0, y_1 \in C^\omega$, if $u(y_0) \le u(y_1)$ then $u(xy_0) \le u(xy_1)$,
(C2) for all $x \in C^+, y \in C^\omega$, $\min\{u(x^\omega), u(y)\} \le u(xy) \le \max\{u(x^\omega), u(y)\}$,
(C3) for any infinite sequence $x_i \in C^+$, $i \in \mathbb{N}$, of non empty colour words such that the infinite word $x_0x_1x_2\ldots$ is finitely generated

$$\min\{u(x_0x_2x_4\ldots), u(x_1x_3x_5\ldots), \inf_{i\in\mathbb{N}} u(x_i^\omega)\} \le u(x_0x_1x_2x_3\ldots) \le \max\{u(x_0x_2x_4\ldots), u(x_1x_3x_5\ldots), \sup_{i\in\mathbb{N}} u(x_i^\omega)\}. \quad (2)$$

Theorem 1 *Let $u : C^\omega \to \overline{\mathbb{R}}$ be a fairly mixing payoff function. Then both players have optimal positional strategies in any game $\mathbf{G} = (G, u)$ over a finite arena G.*

For the sake of simplicity the following lemma is formulated only for player Max, however it should be clear that analogous characterization holds also for player Min. This point will be discussed briefly in the sequel.

Lemma 1 *Let $\mathbf{G} = (G, u)$ be a game with fairly mixing payoff function u. Suppose that*

(1) there exists a vertex $v \in V_{\mathrm{Max}}$ such that $|vE| > 1$, where $vE = \{e \in E \mid \mathrm{source}(e) = v\}$ is the set of all edges with the source v,
(2) $vE = A_1 \cup A_2$ is a partition of the set vE onto two non empty sets A_1 and A_2,
(3) $E' = \{e \in E \mid \mathrm{source}(e) \ne v\}$, $E_1 = E' \cup A_1$, $E_2 = E' \cup A_2$, and $G_1 = (V_{\mathrm{Max}}, V_{\mathrm{Min}}, E_1, C, \varphi)$ and $G_2 = (V_{\mathrm{Max}}, V_{\mathrm{Min}}, E_2, C, \varphi)$ are arenas obtained from G by removing all edges not belonging to E_1 and E_2 respectively (but keeping all vertices V),
(4) players Max *and* Min *have optimal positional strategies in the games $\mathbf{G}_1 = (G_1, u)$ and $\mathbf{G}_2 = (G_2, u)$.*

Then Max *and* Min *have optimal strategies $\sigma^\sharp$ and $\tau^\sharp$ in the game $\mathbf{G} = (G, u)$. More exactly, we can assume without loss of generality that*

$$\mathrm{val}(\mathbf{G}_2(v)) \le \mathrm{val}(\mathbf{G}_1(v)) \quad (3)$$

and then

(i) the optimal positional strategy of player Max *in* $\mathbf{G}_1$ *is also optimal for the same player in the game* $\mathbf{G}$*,*

(ii) for all $w \in V$*,* $\mathrm{val}(\mathbf{G}(w)) = \mathrm{val}(\mathbf{G}_1(w))$*.*

Proof. Let $\sigma_i^\sharp$, $\tau_i^\sharp$ be optimal positional strategies of players Max and Min respectively in the games $\mathbf{G}_i = (G_i, u)$, $i = 1, 2$. We then set

$$\sigma^\sharp = \sigma_1^\sharp \tag{4}$$

It is clear that $\sigma^\sharp$ is a positional strategy for Max not only in the game $\mathbf{G}_1$ but also in the game $\mathbf{G}$. We shall show that $\sigma^\sharp$ is optimal for Max in the game $\mathbf{G} = (G, u)$ and that for all vertices $w \in V$, $\mathrm{val}(\mathbf{G}(w)) = \mathrm{val}(\mathbf{G}_1(w))$

Note the following remark holding under the hypotheses of Lemma 1 and conditions (3) and (4)

Remark 1. Let τ be any strategy of Min in the game $\mathbf{G}$ and w an initial vertex. Then $u(p^G(w, \sigma^\sharp, \tau)) \geq \mathrm{val}(\mathbf{G}_1(w))$, i.e. for games starting at w the strategy $\sigma^\sharp$ can assure to player Max the payoff of at least $\mathrm{val}(\mathbf{G}_1(w))$.

Indeed, if we restrict τ to finite paths in G_1 we obtain a strategy τ_1 of Min in the game $\mathbf{G}_1$. Then the plays in $\mathbf{G}$ consistent with $\sigma^\sharp$ and τ are the same as the plays in $\mathbf{G}_1$ consistent with $\sigma_1^\sharp$ and τ_1, which proves Remark 1.

To finish the proof of Lemma 1 we should construct a strategy $\tau^\sharp$ for player Min assuring that for any strategy σ of player Max in $\mathbf{G}$ and any initial vertex w

$$u(\varphi(p^G(w, \sigma, \tau^\sharp))) \leq \mathrm{val}(\mathbf{G}_1(w)). \tag{5}$$

We define first a mapping

$$b : P_G^* \to \{1, 2\} \tag{6}$$

Let $p \in P_G^*$ be a finite path in G. Set

$$b(p) = \begin{cases} 1 & \text{if either } p \text{ does not contain any edge with the source } v \text{ or} \\ & \text{the last edge of } p \text{ with the source } v \text{ belongs to } G_1, \\ 2 & \text{if the last edge of } p \text{ with the source } v \text{ belongs to } G_2. \end{cases} \tag{7}$$

Then the strategy $\tau^\sharp$ of Min in $\mathbf{G}$ is defined as

$$\tau^\sharp(p) = \begin{cases} \tau_1^\sharp(\mathrm{target}(p)) & \text{if } b(p) = 1 \\ \tau_2^\sharp(\mathrm{target}(p)) & \text{if } b(p) = 2 \end{cases} \tag{8}$$

In other words, playing in $\mathbf{G}$ player Min applies either his optimal strategy $\tau_1^\sharp$ from the game $\mathbf{G}_1$ or his optimal strategy $\tau_2^\sharp$ from the game $\mathbf{G}_2$. Initially, up to the first visit to v, player Min uses the strategy $\tau_1^\sharp$. After the first visit at v the choice between $\tau_1^\sharp$ and $\tau_2^\sharp$ depends on whether the last time when visiting v his adversary Max chose an outgoing edge in E_1 or an edge in E_2. Intuitively, if the last time at v player Max chose a outgoing edge from E_1 then it means that the play from this moment is like a play in $\mathbf{G}_1$ thus player Min tries to respond with his optimal strategy from $\mathbf{G}_1$. Symmetrically, if during the last visit at v player Max chose an outgoing edge from E_2 then from this moment onward the play is like a play in $\mathbf{G}_2$ and player Min tries to counter with his optimal strategy from $\mathbf{G}_2$.

It should be clear that the strategy $\tau^\sharp$ needs in fact just two valued memory $\{1, 2\}$ for player Min to remember if during the last visit to v an edge of E_1 or an edge of E_2 was chosen by his adversary. This memory is initialized to 1 and updated only when vertex v is visited.

In the sequel we shall say that a finite or infinite path p in G is *homogeneous* if one of the following three conditions folds: (1) p never visits v or (2) each edge e of p with source v belongs to E_1 or (3) each edge e of p with source v belongs to E_2.

The proof of (5) is divided on four cases.

Case 1: $w = v$ and the memory state of player Min **is ultimately constant during the play $p = p^G(v, \sigma, \tau^\sharp)$.**
This means that p can be factorized as

$$p = p_0 p_1 ... p_n q$$

where p_i are finite non empty homogeneous paths such that $\text{source}(p_i) = \text{target}(p_i) = v$ and q is an infinite homogeneous path with source v.

Since p is consistent with $\tau^\sharp$ and p_i are homogeneous, each infinite play $p_i^\omega = p_i p_i p_i \ldots$ is either consistent with $\tau_1^\sharp$ (if p_i contains only edges of G_1) or with $\tau_2^\sharp$ (if p_i contains only edges of G_2).

By optimality of strategies $\tau_1^\sharp$ and $\tau_2^\sharp$ we get that either $u(\varphi(p_i^\omega)) \leq \text{val}(\mathbf{G}_1(v))$ or $u(\varphi(p_i^\omega)) \leq \text{val}(\mathbf{G}_2(v))$. Thus, by (3),

$$\text{for all } i,\ 0 \leq i \leq n, \quad u(\varphi(p_i^\omega)) \leq \text{val}(\mathbf{G}_1(v)). \tag{9}$$

Similarly, the infinite path q is either consistent with $\tau_1^\sharp$ or with $\tau_2^\sharp$ implying that $u(\varphi(q))$ cannot be greater than either $\text{val}(\mathbf{G}_1(v))$ or $\text{val}(\mathbf{G}_2(v))$. Thus, again by (3),

$$u(\varphi(q)) \leq \text{val}(\mathbf{G}_1(v)) \tag{10}$$

From (C2) of Definition 1, by a trivial induction on n, we can deduce that $u(x_0 \ldots x_n y) \leq \max\{u(x_0^\omega), \ldots, u(x_n^\omega), u(y))\}$ for any $x_i \in C^+$ and $y \in C^\omega$.

This inequality and (9), (10) imply

$$u(\varphi(p)) = u(\varphi(p_0) \ldots \varphi(p_n)\varphi(q)) \leq \\ \max\{u(\varphi(p_0)^\omega), \ldots, u(\varphi(p_n)^\omega), u(\varphi(q))\} \leq \text{val}(\mathbf{G}_1(v)).$$

Case 2: $w = v$ and the memory state of player Min **changes infinitely often during the play $p = p^G(v, \sigma, \tau^\sharp)$.**

In other words this means that p can be factorized as

$$p = p_0 p_1 p_2 \ldots$$

where, for all i,

(1) p_i is a homogeneous non empty path with source and target v,
(2) the path $p_i p_{i+1}$ is not homogeneous, i.e. if the first edge of p_i is in E_1 then the first edge of the next subpath p_{i+1} is in E_2 and vice versa.

The conditions above imply that the infinite paths $q_0 = p_0 p_2 p_4 \ldots$ and $q_1 = p_1 p_3 p_5 \ldots$ are homogeneous and one of them, say q_0, is consistent with $\tau_1^\sharp$ while the other, say q_1, is consistent with $\tau_2^\sharp$ (in the case where q_0 is consistent with $\tau_2^\sharp$ while q_1 is consistent with $\tau_1^\sharp$ we proceed analogously).

By optimality of $\tau_1^\sharp$ and $\tau_2^\sharp$, $u(q_0) \leq \mathrm{val}(\mathbf{G}_1(v))$ and $u(q_1) \leq \mathrm{val}(\mathbf{G}_2(v))$, implying by (3)

$$u(\varphi(p_0 p_2 p_4 \ldots)) \leq \mathrm{val}(\mathbf{G}_1(v)) \quad \text{and} \quad u(\varphi(p_1 p_3 p_5 \ldots)) \leq \mathrm{val}(\mathbf{G}_1(v)).$$

Note also that each p_i^ω is not only homogeneous but also consistent either with $\tau_1^\sharp$ or with $\tau_2^\sharp$, thus as previously by optimality of both strategies and (3), we get $u(\varphi(p_i)^\omega) \leq \mathrm{val}(\mathbf{G}_1(v)$.

Therefore using condition (C3) of Definition 1 and the bounds established above we get

$$\begin{aligned} u(\varphi(p)) = u(\varphi(p_0)\varphi(p_1)\ldots) \leq \\ \max\{u(\varphi(p_0 p_2 p_4 \ldots)), u(\varphi(p_1 p_3 p_5 \ldots)), \sup_{i \in \mathbb{N}} u(\varphi(p_i)^\omega)\} \\ \leq \mathrm{val}(\mathbf{G}_1(v)). \end{aligned} \tag{11}$$

Case 3: $w \neq v$ **and the play** $p = p^G(w, \sigma, \tau^\sharp)$ **never visits the vertex** v**.**

If we set σ_1 to be the restriction of σ to the paths in G_1 then σ_1 is a strategy of Max in G_1. Moreover since v is never visited player Min using $\tau^\sharp$ in fact applies always the strategy $\tau_1^\sharp$ optimal for him in $\mathbf{G}_1$. Thus p can be seen as a play in G_1 consistent with σ_1 and with $\tau_1^\sharp$ and by optimality of $\tau_1^\sharp$ we get $u(\varphi(p)) \leq \mathrm{val}(\mathbf{G}_1(w))$.

Case 4: $w \neq v$ **and the play** $p = p^G(w, \sigma, \tau^\sharp)$ **visits at least once the vertex** v**.**

Let us factorize p: $p = rq$, where r is the finite prefix of p until the first visit to v, i.e. r is the shortest prefix of p with $\mathrm{target}(r) = v$.

Let $q^\sharp = p^{G_1}(v, \sigma_1^\sharp, \tau_1^\sharp)$, where $\sigma_1^\sharp$ is the optimal positional strategy of Max in $\mathbf{G}_1$. Thus since both $\sigma_1^\sharp$ and $\tau_1^\sharp$ are optimal in $\mathbf{G}_1$ we have $u(q^\sharp) = \mathrm{val}(\mathbf{G}_1(v))$.

Now note that q is in fact a play in G starting at v and consistent with $\tau^\sharp$. This situation was already examined above (case 1 and case 2) and we have learned there that $u(\varphi(q)) \leq \mathrm{val}(\mathbf{G}_1(v))$.

In this way we have obtained

$$u(\varphi(q)) \leq \mathrm{val}(\mathbf{G}_1(v)) = u(\varphi(q^\sharp)) \tag{12}$$

which by condition (C1) of fairly mixing payoffs yields

$$u(\varphi(r)\varphi(q)) \leq u(\varphi(r)\varphi(q^\sharp)) = u(\varphi(rq^\sharp)). \tag{13}$$

However, $rq^\sharp$ is an infinite play in G_1 starting at w and consistent with $\tau_1^\sharp$ (r is consistent with $\tau_1^\sharp$ since until the first visit to v in $p = rq$ player Min plays according to $\tau_1^\sharp$ while $q^\sharp$ is consistent with $\tau_1^\sharp$ just by definition). Thus by optimality of $\tau_1^\sharp$ in $\mathbf{G}_1$ we get $u(\varphi(rq^\sharp)) \leq \mathrm{val}(\mathbf{G}_1(w))$. This and (13) imply $u(\varphi(p)) = u(\varphi(rq)) \leq \mathrm{val}(\mathbf{G}_1(w))$.

This concludes the proof of Lemma 1.

□

Before applying Lemma 1 let us note what happens if all hypothesis of Lemma 1 are satisfied except that the vertex v belongs rather to Min. Suppose also that, as in Lemma 1, $\mathrm{val}(\mathbf{G}_2(v)) \leq \mathrm{val}(\mathbf{G}_1(v))$. Then it is the optimal strategy $\tau_2^\sharp$ of Min from $\mathbf{G}_2$ that is optimal in $\mathbf{G}$. This can be deduced immediately from Lemma 1 since player Min can be seen as the maximizer of the payoff $-u$.

Proof of Theorem 1

Let $G = (V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, C, \varphi)$ be a finite arena and $\mathbf{G} = (G, u)$ with fairly mixing payoff u.

We prove the theorem by induction on $n_G = |E| - |V|$.

If $n_G = 0$ then, since G has no dead ends, each vertex of V has only one outgoing edge. Thus the players have no choice and there is only one possible strategy for each of them and these strategies are positional. Obviously they are also optimal.

Let $n_G > 0$ and suppose that the thesis holds for each game $\mathbf{G}'$ over an arena such that $n_{G'} < n_G$.

If all vertices $v \in vM$ of player Max have only one outgoing edge then Max has only one possible strategy and this strategy is positional. Obviously, this unique strategy is also optimal for Max.

Now suppose that there exists $v \in vM$ having at least two outgoing edges. We decompose G onto two subarenas G_1 and G_2 exactly as in Lemma 1. Since G_1 and G_2 have the same number of vertices as G but their number of edges is strictly less than $|E|$ we can apply to $\mathbf{G}_i = (G_i, u)$, $i = 1, 2$, the induction hypothesis to deduce the existence of optimal positional strategies $\sigma_i^\sharp$ for Max in $\mathbf{G}_i$, $i = 1, 2$. Again by Lemma 1, either $\sigma_1^\sharp$ or $\sigma_2^\sharp$ is an optimal positional strategy of Max in $\mathbf{G}$, depending on whether $\mathrm{val}(\mathbf{G}_2(v)) \leq \mathrm{val}(\mathbf{G}_1(v))$ or the inverse inequality holds.

The existence of an optimal positional strategy for player Min follows by a symmetric argument.

□

4 Applications

In this section we show that virtually all popular (as well as many less popular) payoff mappings satisfy conditions (C1)-(C3). This implies immediately the existence of positional optimal strategies due to Theorem 1. Due to space restrictions all proofs are relegated to Appendix.

If not stated explicitly otherwise, in the examples examined below we suppose that $C = \mathbb{R}$, i.e. the edges are labeled by real numbers. In particular, $\mathbb{R}^+$ will stand always for the set of non empty finite sequences of real numbers and $\mathbb{R}^\omega$ is the set of finitely generated infinite sequences of reals.

Sup game. Max wins the highest value seen during the play, i.e. the payoff is

$$u_s(c_0c_1\ldots) = \sup\{c_0, c_1, \ldots\}, \quad \text{where } c_i \in C.$$

Limsup game. Now we suppose that Max wins the highest value seen infinitely often during the play, i.e. the payoff is given by

$$u_l(c_0c_1\ldots) = \limsup_i c_i.$$

This payoffs are used for example in gambling systems.

Total reward game. In the total reward game player Max accumulates the payoffs :

$$u_t(c_0c_1\ldots) = \limsup_n \sum_{i=0}^{n} c_i.$$

Note that in this case the payoff can take infinite values $\pm\infty$. This type of payoffs is classical in game theory [6].

Parity game. $C = \mathbb{N}$ is the set of non negative integers. The payoff is defined as

$$u_p(c_0c_1\ldots) = (\limsup_{i\in\mathbb{N}} c_i) \mod 2$$

In other words, player Max wins 1 if the highest colour visited infinitely often is odd, otherwise his payoff is 0. This is the most relevant type of payoff for computer science, [5].

Weighted reward game. $C = \mathbb{R}$ is again the set of real numbers. The payoff is given by

$$u_m^\lambda(c_0c_1...) = \lambda \cdot \liminf_{i\in\mathbb{N}} c_i + (1-\lambda) \cdot \limsup_{i\in\mathbb{N}} c_i,$$

where $\lambda \in [0,1]$ is any fixed constant from the closed interval $[0,1]$.

The interpretation for u_m^λ is the following. If c_i is the capital of player Max on the day i then using the coefficient λ he can weight relatively his "good" and "bad" days. If he is optimistic then he will use $\lambda = 1$ and take into account only the good days when his capital is maximal, this is the play considered in Section 6.2. If he is pessimistic then he will set $\lambda = 0$ and play in such a way as his "bad" days were not too bad.

With λ between 0 and 1 he carefully weights relative pleasure of the happy days against the difficulties of more harsh times.

Note also that the payoff $u = 2 \cdot u_m^{\frac{1}{2}}$ can be seen as a generalization of the parity payoff. To see this let us take a parity game $\mathbf{G}$ with an underlying arena G. Let us replace in G each odd label c by $-c$ and now consider over this modified arena the game $\mathbf{G}' = (G', u)$ with the payoff u defined above. Now it suffices to note that if the game value of $\mathbf{G}'(s)$ is non negative then in the original parity game $\mathbf{G}$ it is the player 0 that wins. On the other hand, if the game value of $\mathbf{G}'(s)$ is negative then in the original parity game $\mathbf{G}$ it is the player 1 that wins. The game $\mathbf{G}'$ can be seen as quantitative version of the parity game. In the parity game we examine only if the maximal infinitely often visited colour is even or odd, in the game $\mathbf{G}'$ we measure more precisely the distance between the greatest even and odd colours visited infinitely often.

Mean payoff game. Again $C = \mathbb{R}$. With any finite sequence $x \in \mathbb{R}^+$ of elements of $\mathbb{R}$ we associate their mean value

$$\mathrm{mean}(x) = \frac{1}{|x|}\mathfrak{S}(x),$$

where like in Section 6.3 $\mathfrak{S}(x)$ denotes the sum of all elements of x while $|x|$ stands for the length of x. The mean payoff mapping is defined by

$$u_m(c_0c_1\dots) = \limsup_{n\in\mathbb{N}}(\mathrm{mean}(c_0\dots c_{n-1}))$$

Discounted Game. The set of colours is $C = [0,1) \times \mathbb{R}$.

For any finitely generated infinite word $(\lambda_0, a_0)(\lambda_1, a_1)\ldots \in C^\omega$ we set

$$u_d((\lambda_0,a_0)(\lambda_1,a_1)\ldots) = \lambda_0 a_0 + \lambda_0\lambda_1 a_1 + \lambda_0\lambda_1\lambda_2 a_2 + \cdots$$

Usually in discounted games there is one discount factor λ and then $u_d(a_0a_1\ldots) = \sum_{i\geq 0}\lambda^i a_i$. Allowing different discout factors is in the spirit of the original paper of Shapley [9].

Theorem 2 *All payoff mappings listed above satisfy conditions (C1)-(C2) and therefore both players have optimal positional strategies.*

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Appendix

5 Special payoff functions

In this section we examine some special useful classes of payoff functions.

5.1 Prefix independent payoffs

Some well-known payoffs functions, such as parity payoff or mean-payoff do not depend on any finite prefix of the play.

A payoff mapping u is said to be *prefix independent* if for any $x \in C^+$ and $y \in C^\omega$, $u(y) = u(xy)$. The trivial proof of the following remark is left to the reader.

Remark 2. If u is prefix independent then u satisfies conditions (C1) and (C2) of Definition 1 of fairly mixing payoffs.

Thus to prove that prefix independent u is fairly mixing it suffices to verify (C3).

5.2 Locally continuous payoffs

We shall assume here the usual metric over C^ω: for two words $c_0c_1\ldots$ and $d_0d_1\ldots$ of C^ω

$$d(c_0c_1\ldots,d_0d_1\ldots)=2^{-\min\{n\in\mathbb{N}|c_n\neq d_n\}},$$

see [8] for further details about the topology on C^ω. It turns out that continuous payoff mappings $u:C^\omega\to\mathbb{R}$ allow simpler proofs of fairly mixing property. In fact what we need to this end is even weaker property of local continuity:

Definition 2. *A payoff $u:C^\omega\to R$ is said to be locally continuous if, for any finite subset $D\subset C$, the restriction $u_{|D^\omega}:D^\omega\to\mathbb{R}$ is continuous.*

We can now state precisely our result about locally continuous payoffs.

Lemma 2 *If a locally continuous payoff $u:C^\omega\to\mathbb{R}$ satisfies conditions (C1) and (C2) of fairly mixing, then u is a fairly mixing payoff.*

Proof. Let $u:C^\omega\to\mathbb{R}$ be a payoff satisfying the hypothesis of Lemma 2. We prove that condition (C3) is satisfied. Let $x_n\in C^+$, $n\in\mathbb{N}$, be a sequence of finite non empty colour words such that $x_0x_1x_2\ldots$ is finitely generated. Suppose that

$$\forall i\in\mathbb{N},u(x_0x_1x_2\ldots)>u(x_i^\omega). \tag{14}$$

We shall prove that

$$u(x)\le u(x_0x_2x_4\ldots) \tag{15}$$

which immediately yields the second inequality of (C3).

First, we prove by induction the following

Remark 3. The sequence $(u(x_nx_{n+1}\ldots))_{n\in\mathbb{N}}$, is non decreasing.

Suppose, by induction hypothesis, that the remark holds for the first n elements of the sequence, i.e.

$$u(x_0x_1\ldots)\le u(x_1x_2\ldots)\le\ldots\le u(x_nx_{n+1}\ldots).$$

This and (14) imply that $u(x_n^\omega)<u(x_nx_{n+1}\ldots)$. However condition (C2) of fairly mixing tells us that $u(x_nx_{n+1}\ldots)\le\max\{u(x_n^\omega),u(x_{n+1}x_{n+2}\ldots)\}$. The last two inequalities imply $u(x_nx_{n+1}\ldots)\le u(x_{n+1}x_{n+2}\ldots)$, which achieves the proof of Remark 3.

From this remark $u(x_{2n+1}x_{2n+2}\ldots)\le u(x_{2n+2}x_{2n+3}\ldots)$ and by condition (C1) of fairly mixing we can append to the words in this inequality the prefix $x_0x_2x_4\ldots x_{2n}$. This yields

$$u(x_0x_2x_4\ldots x_{2n}x_{2n+1}x_{2n+2}x_{2n+3}\ldots)\le u(x_0x_2x_4\ldots x_{2n}x_{2n+2}x_{2n+3}\ldots)$$

i.e. the sequence $u(x_0x_2x_4\ldots x_{2n}x_{2n+1}x_{2n+2}\ldots),n\in\mathbb{N}$, is also non decreasing. As the first element of this sequence is $u(x_0x_1x_2\ldots)$, we get

$$u(x_0x_1x_2\ldots)\le u(x_0x_2x_4\ldots x_{2n}x_{2n+1}x_{2n+2}x_{2n+3}\ldots),\quad\text{for all } n\in\mathbb{N}. \tag{16}$$

Since $x_0x_1x_2\ldots$ is finitely generated, there exists a finite subset $D \subset C$ such that every letter of $x_0x_1x_2\ldots$ is an element of D. Note that the sequence $(x_0x_2x_4\ldots x_{2n}x_{2n+1}x_{2n+2}\ldots)_{n\in\mathbb{N}}$ converges to $x_0x_2x_4\ldots$ in D^ω when n goes to ∞. Since u is locally continuous, $(u(x_0x_2x_4\ldots x_{2n}x_{2n+1}x_{2n+2}\ldots))_{n\in\mathbb{N}}$ converges to $u(x_0x_2x_4\ldots)$. Together with (16), it proves by continuity of u that $u(x_0x_1x_2\ldots) \leq u(x_0x_2x_4\ldots)$. The proof of second inequality of (C3) is accomplished.

For proving the first inequality of (C3), it suffices to consider the payoff $-u$ and note that the first inequality of (C3) for u is the same as the second inequality for $-u$. □

6 Applications.

If not stated explicitly otherwise, in the examples examined below we suppose that $C = \mathbb{R}$, i.e. the edges are labeled by real numbers. In particular, $\mathbb{R}^+$ will stand always for the set of non empty finite sequences of real numbers and $\mathbb{R}^\omega$ is the set of finitely generated infinite sequences of reals.

6.1 Sup game.

Max wins the highest value seen during the play, i.e. the payoff is

$$u_s(c_0c_1\ldots) = \sup\{c_0, c_1, \ldots\}, \quad \text{where } c_i \in C.$$

Since $u_s(xy) = \max\{u_s(x^\omega), u_s(y)\}$, for $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^\omega$, conditions (C1) and (C2) of fairly mixing payoff are satisfied immediately. On the other hand, the equality $u_s(x_0x_1\ldots) = \sup\{u_s(x_0^\omega), u_s(x_1^\omega), \ldots\}$, for any $x_i \in \mathbb{R}^+, i \in \mathbb{N}$, implies condition (C3).

It is obvious that the payoff remains fairly mixing if Max wins rather the minimal value seen during the game : $u_i(c_0c_1\ldots) = \inf\{c_0, c1, \ldots\}$.

6.2 Limsup game.

Now we suppose that Max wins the highest value seen infinitely often during the play, i.e. the payoff is given by

$$u_l(c_0c_1\ldots) = \limsup_i c_i.$$

Since this payoff mapping is prefix independent, by Remark 2, it suffices to verify condition (C3) of fairly mixing payoffs. However, this is immediate since

$$u_l(x_0x_1\ldots) = \limsup_i u_l(x_i^\omega)$$

for any sequence (x_i), where $x_i \in C^+$.

Again in the definition of the payoff we can replace lim sup by lim inf and the proof remains similar.

6.3 Total reward game.

In the total reward game player Max accumulates the payoffs :

$$u_t(c_0c_1\ldots) = \limsup_n \sum_{i=0}^{n} c_i.$$

Note that in this case the payoff can take infinite values $\pm\infty$.

We shall verify that u_t is fairly mixing.

For $x \in \mathbb{R}^+$ we shall note by $\mathfrak{S}(x)$ the sum of elements of the real sequence x. With this notation we note the following equality :

$$\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^\omega, u_t(xy) = \mathfrak{S}(x) + u_t(y) \tag{17}$$

where we adopted the natural convention that $a + \infty = \infty$ and $a - \infty = -\infty$ for $a \in \mathbb{R}$. By trivial case inspection this yield condition (C1) of fairly mixing payoffs.

The following implications hold for any $x \in \mathbb{R}^+$:

$$\begin{array}{llll} \text{if} & \mathfrak{S}(x) < 0 & \text{then} & u_t(x^\omega) = -\infty, \\ \text{if} & \mathfrak{S}(x) = 0 & \text{then} & u_t(x^\omega) \in \mathbb{R}, \\ \text{if} & \mathfrak{S}(x) > 0 & \text{then} & u_t(x^\omega) = +\infty. \end{array} \tag{18}$$

Indeed, let $x' \sqsubseteq x$ be the prefix of x such that $\mathfrak{S}(x')$ is maximal (we can take the shortest such prefix if there are several such prefixes). Then $u_t(x^\omega) = \limsup_i \mathfrak{S}(x^i x') = \limsup_i (i \cdot \mathfrak{S}(x) + \mathfrak{S}(x')) = \lim_{i\to\infty} \sup_{j\geq i}(j \cdot \mathfrak{S}(x) + \mathfrak{S}(x'))$. However

$$\sup_{j\geq i}(j \cdot \mathfrak{S}(x) + \mathfrak{S}(x')) = \begin{cases} -\infty & \text{if } \mathfrak{S}(x) < 0 \\ \mathfrak{S}(x') & \text{if } \mathfrak{S}(x) = 0 \\ \infty & \text{if } \mathfrak{S}(x) > 0 \end{cases}$$

and we get (18).

Now note that condition (C2) of fairly mixing payoffs reads for u_t as

$$\min\{u_t(x^\omega), u_t(y)\} \leq \mathfrak{S}(x) + u_t(y) \leq \max\{u_t(x^\omega), u_t(y)\}$$

which can be verified readily by elementary case analysis with the help of (18).

It remains to verify condition (C3) of fairly mixing payoffs. We give first a proof for the second inequality in (C3):

$$u(x_0x_1x_2x_3\ldots) \leq \max\{u(x_0x_2x_4\ldots), u(x_1x_3x_5\ldots), \inf_{i\in\mathbb{N}} u(x_i^\omega)\} \tag{19}$$

where $x_i \in \mathbb{R}^+, i = 0, 1, 2, \ldots$.

If there exists $n \in \mathbb{N}$ such that $\mathfrak{S}(x_n) > 0$ then $u_t(x_n^\omega) = +\infty$ and (19) holds.

In the opposite case, $\forall n \in \mathbb{N}, \mathfrak{S}(x_n) \leq 0$. Then

$$\forall n \in \mathbb{N}, \sum_{0\leq i\leq n} \mathfrak{S}(x_i) \leq \sum_{\substack{0\leq i\leq n \\ i \text{ even}}} \mathfrak{S}(x_i).$$

By taking the $\limsup_{n\in\mathbb{N}}$ of this inequality, we obtain $u_t(x_0x_1\ldots) \leq u_t(x_0x_2\ldots)$ and (19) holds again.

The proof of the first inequality in (C3) can be carried in exactly the same way, except that we should consider first the case where there exists $n \in \mathbb{N}$ such that $\mathfrak{S}(x_n) < 0$ and next the opposite case.

In this way we have shown that u_t defines fairly mixing payoff.

6.4 Parity game.

$C = \mathbb{N}$ is the set of non negative integers. The payoff is defined as

$$u_p(c_0c_1\ldots) = (\limsup_{i\in\mathbb{N}} c_i) \mod 2$$

In other words, player Max wins 1 if the highest colour visited infinitely often is odd, otherwise his payoff is 0. This payoff mapping is prefix independent thus it suffices to check if condition (C3) of fairly mixing payoffs holds. However this is rather trivial.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathbb{N}^+$ such that $x_0x_1x_2\ldots$ is finitely generated. Clearly,

$$u_p(x_0x_1x_2\ldots) = \limsup_{n\in\mathbb{N}} u_p(x_n^\omega)$$

Since $x_0x_1x_2\ldots$ is finitely generated, the set $\{u_p(x_n^\omega) : n \in \mathbb{N}\}$ is finite and there exists $n \in \mathbb{N}$ such that $u_p(x_0x_1x_2\ldots) = u_p(x_n)$. This immediately implies (C3).

6.5 Weighted reward game.

$C = \mathbb{R}$ is again the set of real numbers. The payoff is given by

$$u_m^\lambda(c_0c_1...) = \lambda \cdot \liminf_{i\in\mathbb{N}} c_i + (1-\lambda) \cdot \limsup_{i\in\mathbb{N}} c_i,$$

where $\lambda \in [0,1]$ is any fixed constant from the closed interval $[0,1]$.

It is a prefix independent payoff. It suffices to check (C3). For the sake of simplicity let us suppose that $(x_i)_{i\in\mathbb{N}}$ is a finitely generated sequence of elements of $\mathbb{R}$ (the property can be proved without assuming that (x_i) is finitely generated but the proof is a bit more complicated).

Let c_i be the i-th real number in the sequence $x = x_0x_1x_2\ldots$. Since x is finitely generated there exists n such that for all $k \geq n$ $\liminf_i c_i \leq c_k \leq \limsup_i c_i$ and for infinitely many $k \geq n$, $\liminf_i c_i = c_k$ as well as for infinitely many $l \geq n$, $c_l = \limsup_i c_i$. This shows that there exists elements x_k and x_l in the sequence of words (x_i) such that

$$\liminf_i c_i = \min(x_k) \leq \max(x_k) \leq \limsup_i c_i$$

and

$$\liminf_i c_i \leq \min(x_l) \leq \max(x_l) = \limsup_i c_i.$$

However, this implies respectively that

$$u_m^\lambda(x_k^\omega) \leq u_m^\lambda(x) \quad \text{and} \quad u_m^\lambda(x_l^\omega) \geq u_m^\lambda(x)$$

which implies in turn (3).

6.6 Mean payoff game.

Again $C = \mathbb{R}$. With any finite sequence $x \in \mathbb{R}^+$ of elements of $\mathbb{R}$ we associate their mean value

$$\mathrm{mean}(x) = \frac{1}{|x|}\mathfrak{S}(x),$$

where like in Section 6.3 $\mathfrak{S}(x)$ denotes the sum of all elements of x while $|x|$ stands for the length of x. The mean payoff mapping is defined by

$$u_m(c_0c_1\dots) = \limsup_{n\in\mathbb{N}}(\mathrm{mean}(c_0\dots c_{n-1}))$$

Again, since this payoff is prefix independent we have just to prove (C3).

Let $x_i \in \mathbb{R}^+, i = 0, 1, 2\dots$ and let us set

$$\begin{aligned} \mathbf{z} &= x_0x_1x_2\dots \\ \mathbf{z}^{\mathrm{even}} &= x_0x_2x_4\dots \\ \mathbf{z}^{\mathrm{odd}} &= x_1x_3x_5\dots \end{aligned} \tag{20}$$

We shall write $\mathbf{z}[i], \mathbf{z}^{\mathrm{even}}[i], \mathbf{z}^{\mathrm{odd}}[i], i = 0, 1, \dots$ to denote the i-th real number in each of the three infinite real sequences.

We begin with proving the first inequality of (C3). Since $(x_0\dots x_n)_{n\in\mathbb{N}}$ is a subsequence of $(\mathbf{z}[0]\dots\mathbf{z}[n])_{n\in\mathbb{N}}$,

$$\begin{aligned} u_m(x_0x_1x_2\dots) &= \limsup_{n\in\mathbb{N}}(\mathrm{mean}(\mathbf{z}[0]\dots\mathbf{z}[n])) \\ &\geq \limsup_{n\in\mathbb{N}}((\mathrm{mean}(x_0\dots x_n))), \end{aligned} \tag{21}$$

Since

$$\mathrm{mean}(x_0\dots x_n) = \frac{1}{|x_0\dots x_n|}\sum_{0\leq i\leq n}|x_i|\,\mathrm{mean}(x_i),$$

$\mathrm{mean}(x_0\dots x_n)$ is a convex combination of $\{\mathrm{mean}(x_0),\dots,\mathrm{mean}(x_n)\}$. Therefore

$$\mathrm{mean}(x_0\dots x_n) \geq \min\{\mathrm{mean}(x_i) \mid 0 \leq i \leq n\} \geq \inf\{\mathrm{mean}(x_i) \mid i \in \mathbb{N}\}.$$

Together with (21), it implies

$$u_m(x_0x_1x_2\dots) \geq \inf\{\mathrm{mean}(x_i) \mid i \in \mathbb{N}\}.$$

Since $\mathrm{mean}(x_i) = u_m(x_i^\omega)$, we obtain the first inequality of (C3).

Now we prove the second inequality of (C3). Let A_0 be the set of all integers i such that $\mathbf{z}[i]$ belongs to a factor x_{2k} of an even index in the factorization (20) and let A_1 be the set of all integers i such that $\mathbf{z}[i]$ belongs to a factor x_{2k+1} of an odd index in the factorization (20).

Obviously $\mathbb{N} = A_0 \cup A_1$ is a factorization of $\mathbb{N}$ and we have

$$\begin{aligned}\operatorname{mean}(\mathbf{z}[0]\dots\mathbf{z}[n]) &= \frac{1}{n}\Big(\sum_{\substack{0\le i<n\\ i\in A_0}} \mathbf{z}[i] + \sum_{\substack{0\le i<n\\ i\in A_1}} \mathbf{z}[i]\Big)\\ &= \frac{n_0}{n}\cdot\Big(\frac{1}{n_0}\cdot\sum_{\substack{0\le i<n\\ i\in A_0}} \mathbf{z}_i\Big) + \frac{n_1}{n}\cdot\Big(\frac{1}{n_1}\cdot\sum_{\substack{0\le i<n\\ i\in A_1}} \mathbf{z}[i]\Big)\\ &= \frac{n_0}{n}\cdot\operatorname{mean}(\mathbf{z}^{\text{even}}[0]\dots\mathbf{z}^{\text{even}}[n_0-1]) + \frac{n_1}{n}\cdot\operatorname{mean}(\mathbf{z}^{\text{odd}}[0]\dots\mathbf{z}^{\text{odd}}[n_1-1])\end{aligned} \tag{22}$$

where $n_0 = |A_0 \cap [0..n-1]|$ and $n_1 = |A_1 \cap [0..n-1]|$ are respectively the number of terms in the first and the second sum above. Since $n = n_0 + n_1$ the mean value of n first elements of $\mathbf{z}$ is a convex combination of means of some prefixes of $\mathbf{z}^{\text{even}}$ and $\mathbf{z}^{\text{odd}}$.

This implies for each n,

$$\begin{aligned}&\operatorname{mean}(\mathbf{z}[0]\dots\mathbf{z}[n-1]) \le\\ &\quad\max\{\operatorname{mean}(\mathbf{z}^{\text{even}}[0]\dots\mathbf{z}^{\text{even}}[n_0-1]),\ \operatorname{mean}(\mathbf{z}^{\text{odd}}[0]\dots\mathbf{z}^{\text{even}}[n_1-1])\}\end{aligned} \tag{23}$$

However if $n \to \infty$ then $n_0 \to \infty$ and $n_1 \to \infty$, therefore from (23) we can deduce that

$$\begin{aligned}&u_m(\mathbf{z}) = \limsup_n \operatorname{mean}(\mathbf{z}[0]\dots\mathbf{z}[n-1]) \le\\ &\max\{\limsup_{n_0} \operatorname{mean}(\mathbf{z}^{\text{even}}[0]\dots\mathbf{z}^{\text{even}}[n_0-1]),\ \limsup_{n_1} \operatorname{mean}(\mathbf{z}^{\text{odd}}[0]\dots\mathbf{z}^{\text{odd}}[n_1-1])\} \le\\ &\qquad\qquad\max\{u_m(\mathbf{z}^{\text{even}}), u_m(\mathbf{z}^{\text{odd}})\}\end{aligned}$$

which shows that the second inequality of (C3) holds.

6.7 Discounted Game.

The set of colours is $C = [0,1) \times \mathbb{R}$.

For any finitely generated infinite word $(\lambda_0, a_0)(\lambda_1, a_1)\dots \in C^\omega$ we set

$$u_d((\lambda_0, a_0)(\lambda_1, a_1)\dots) = \lambda_0 a_0 + \lambda_0\lambda_1 a_1 + \lambda_0\lambda_1\lambda_2 a_2 + \cdots$$

The value of u_d is well-defined and finite. Indeed, let $D \subset C$ be a finite alphabet that contains all letters $(\lambda_0, a_0), (\lambda_1, a_1), \dots$. Set

$$\begin{aligned}\lambda_{\max} &= \max\{\lambda \in [0,1) \mid \exists a \in \mathbb{R} \text{ such that } (\lambda, a) \in D\}\\ a_{\max} &= \max\{|a| \in \mathbb{R} \mid \exists \lambda \in [0,1) \text{ such that } (\lambda, a) \in D\}.\end{aligned} \tag{24}$$

Then

$$|\sum_{i=0}^{\infty} \lambda_0 \dots \lambda_i a_i| \leq \sum_{i=0}^{\infty} |\lambda_0 \dots \lambda_i a_i| \leq \sum_{i=0}^{\infty} \lambda_{max}^{i+1} a_{max} = \frac{a_{\max}\lambda_{max}}{1-\lambda_{\max}}.$$

We shall prove that u_d is fairly mixing. Let $x \in C^+$ and $y \in C^\omega$ and let (λ_i, a_i) be the i^{th} letter of x, for $0 \leq i < |x|$. The condition (C1) is immediately implied by the following equality:

$$u_d(xy) = (\sum_{0 \leq i < |x|} \lambda_0 \dots \lambda_i a_i) + \lambda_0 \dots \lambda_{|x|-1} u(y). \tag{25}$$

Moreover, a simple computation yields

$$\begin{aligned} u_d(x^\omega) &= \sum_{n \in \mathbb{N}} (\lambda_0 \dots \lambda_{|x|-1})^n (\sum_{0 \leq i < |x|} \lambda_0 \dots \lambda_i a_i) \\ &= \frac{1}{1 - \lambda_0 \dots \lambda_{|x|-1}} \sum_{0 \leq i < |x|} \lambda_0 \dots \lambda_i a_i \end{aligned}$$

which, together with (25), shows that $u_\lambda(xy)$ is a convex combination of $u_\lambda(x^\omega)$ and $u_\lambda(y)$. It yields immediately (C2).

Now, we prove that $u_d : C^\omega \to \mathbb{R}$ is locally continuous, which, according to Lemma 2, proves that u_d is fairly mixing. Let $D \subset C$ be a finite alphabet and λ_{max} and a_{max} be as in (24). Let $n \in \mathbb{N}$ and $z_0, z_1 \in D^\omega$ such that $d(z_0, z_1) \leq 2^{-n}$. Then z_0 and z_1 share the same prefix $x \in D^*$ of length n and they can be rewritten $z_0 = xy_0$ and $z_1 = xy_1$.

Then

$$\begin{aligned} |u_d(z_0) - u_d(z_1)| &= |\lambda_0 \dots \lambda_{|x|-1}| \cdot |u_d(y_0) - u_d(y_1)| \\ &\leq \lambda_{\max}^n \cdot (|u_d(y_0)| + |u_d(y_1)|) \\ &\leq \lambda_{\max}^{n+1} \cdot \frac{2a_{\max}}{1 - \lambda_{\max}} \end{aligned}$$

Since $0 \leq \lambda_{\max} < 1$, this last term converges to 0 when n tends to $+\infty$, which proves that u_d is continuous.